# Almost Complex and Almost Product Einstein Manifolds from a Variational Principle

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**Abstract**

It is shown that the first order (Palatini) variational principle for a generic nonlinear metric-affine Lagrangian depending on the (symmetrized) Ricci square invariant leads to an almost-product Einstein structure or to an almost-complex anti-Hermitian Einstein structure on a manifold. It is proved that a real anti-Hermitian metric on a complex manifold satisfies the Kähler condition on the same manifold treated as a real manifold if and only if the metric is the real part of a holomorphic metric. A characterisation of anti-Kähler Einstein manifolds and almost-product Einstein manifolds is obtained. Examples of such manifolds are considered.

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# 1 Introduction

Almost-complex and almost-product structures are among the most important geometrical structures which can be considered on a manifold [15, 58, 36]. The aim of this paper is to show that structures of this kind appear in a natural way from a variational principle based on a general class of Lagrangians depending on the Ricci square invariant constructed out of a metric and a symmetric connection; in particular we will show that an anti-Hermitian metric and its special case, so called "anti-Kählerian" metric, appear naturally from our variational principle. Manifolds with such metrics are much less studied than the familiar Hermitian and Kählerian cases. We hope that our variational principle and "the universality of the Einstein equations" (see below) will provide an additional motivation for investigating these manifolds.

Let $M$ be a differentiable manifold of dimension $n$ and $L(M)$ be its frame bundle, a principal fibre bundle over $M$ with group $GL(n;\mathbb{R})$. Let $G$ be a Lie subgroup of $GL(n;\mathbb{R})$. A differentiable subbundle $Q$ of $L(M)$ with structure group $G$ is called a *$G$-structure on $M$* [15, 36]. The classification and integrability of $G$-structures have been studied in differential geometry; algebraic-topological conditions on $M$ which are necessary for the existence of a $G$-structure on $M$ can be given in terms of characteristic classes (see, for example, [54]). We also recall that there is a natural one-to-one correspondence between pseudo-Riemannian metrics of signature $q$ on $M$ and $O(p, q;\mathbb{R})$-structures on $M$, with $p + q = n$. An $O(p, q;\mathbb{R})$-structure is integrable if and only if the corresponding pseudo-Riemannian metric has vanishing Riemann curvature. If $G = GL(p;\mathbb{R}) \times GL(q;\mathbb{R})$ then the $G$-structure is called an *almost-product structure* [58, 47, 59]; if $G = O(r, s;\mathbb{R}) \times O(k, l;\mathbb{R})$ then the $G$-structure is called a *(pseudo-)Riemannian almost-product structure* [58, 55, 30]. If $n$ is even, $n = 2m$, and one considers $GL(m;\mathbb{C})$ as a subgroup of $GL(2m;\mathbb{R})$ then a $GL(m;\mathbb{C})$-structure is called an *almost-complex structure* [58, 36]; if, moreover, one considers $O(m;\mathbb{C})$ as a subgroup of $GL(m;\mathbb{C})$ then an $O(m;\mathbb{C})$-structure defines an *almost-complex anti-Hermitian structure* [43, 25, 14, 26, 5].

We will here use an equivalent description of these $G$-structures. Let $M$ be a manifold and $P$ be an endomorphism of the tangent bundle $TM$ satisfying $P^2 = I$, where $I = identity$. Then $P$ defines an almost-product structure on $M$. If $g$ is a metric on $M$ such that $g(PX, PY) = g(X, Y)$ for arbitrary vectorfields $X$ and $Y$ on $M$, then the triple $(M, g, P)$ defines a (pseudo-) Riemannian almost-product structure. Geometric properties

of (pseudo-) Riemannian almost-product structures have been studied in [58, 55, 30, 42, 27, 13, 49, 53, 47]. If, moreover, $g$ is an Einstein metric (*i.e.* $Ric(g) = \gamma g$ holds, where $Ric(g)$ is the Ricci tensor and $\gamma$ is a constant) then the triple $(M, g, P)$ shall be called an *almost-product Einstein manifold*.

Analogously, if $J$ is an endomorphism of the tangent bundle $TM$ satisfying $J^2 = -I$, then $J$ defines an almost-complex structure on $M$. An almost-complex structure is integrable if and only if it comes from a complex structure (see [37]). If $g$ is a metric on $M$ such that $g(JX, JY) = -g(X, Y)$ for arbitrary vectorfields $X$ and $Y$ on $M$ then the triple $(M, g, J)$ defines an almost-complex anti-Hermitian structure. The metric $g$ in this case is called a *Norden metric* and in complex coordinates it has the form

$$ds^2 = g_{ab}dz^a dz^b + g_{\bar{a}\bar{b}}dz^{\bar{a}}dz^{\bar{b}}$$

where $g_{\bar{a}\bar{b}} = \bar{g}_{ab}$. This canonical form differs from the well known form of an Hermitian metric $ds^2 = 2g_{a\bar{b}}dz^a dz^{\bar{b}}$. We will show (theorem 4.2) that the condition $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection, is equivalent in this case to analyticity of the metric: $\bar{\partial}_c g_{ab} = 0$ . Such anti-Hermitian metrics shall be called *anti-Kählerian* metrics since for an Hermitian metric the condition $\nabla J = 0$ defines a Kählerian metric.

If $g$ is an Einstein metric, *i.e.* $Ric(g) = \gamma g$ holds, then the almost-complex anti-Hermitian manifold $(M, g, J)$ is called an *anti-Hermitian Einstein manifold*. We will consider an important particular class of such manifolds, namely those characterized by *anti-Kählerian Einstein metrics*. Let us stress that we treat *the whole* complex manifold as a real manifold and in this way we get a real Einstein metric with signature $(m, m)$. Another approach to complex Einstein equations, dealing with a *real section* of a complex manifold and aiming to get the Lorentz signature, has been considered in [46, 41, 39, 34, 28, 24].

These $G$-structures can be conveniently defined as a triple $(M, g, K)$, where $g$ is a metric on $M$ and $K$ is a (1,1) tensor field on $M$ such that $K^2 = \epsilon I$ and $g(KX, KY) = \epsilon g(X, Y)$ for arbitrary vectorfields $X$ and $Y$ on $M$ ($\epsilon \neq 0$ is a real constant). We shall also call them *K-structures*. If $\epsilon = 1$ then $K$ defines an almost-product structure on $M$; if $\epsilon = -1$ then $K$ defines an almost-complex structure on $M$. The more general case $\epsilon > 0$ can be reduced to $\epsilon = 1$ by a suitable rescaling, while the case $\epsilon < 0$ is reduced to $\epsilon = -1$. In any coordinate system one has $K^\mu_\alpha K^\alpha_\nu = \epsilon \delta^\mu_\nu$ and $K^t g K = \epsilon g$, where $K^t$ is the transpose matrix, $\mu, \nu, \alpha = 1, 2, ..., n = dim\ (M)$ and $\delta^\mu_\nu$ is the Kronecker symbol.

One can then define a new metric $h$ by the relation $h(X,Y) = g(KX,Y)$, or equivalently $h = gK$, *i.e.* in local coordinates $h_{\mu\nu} = g_{\mu\alpha}K^{\alpha}_{\nu}$; then the following holds

$$(g^{-1}h)^2 = \epsilon I. \tag{1.1}$$

The relation (1.1) for $\epsilon = +1$ or $-1$ is equivalent to $(h^{-1}g)^2 = \epsilon I$ and there is a one-to-one correspondence between the $G$-structure $(M, g, K)$ and the $G$-structure $(M, h, K^{-1})$. Thence the $G$-structure given by the triple $(M, g, K)$ can be equivalently described by the triple $(M, g, h)$, where $g$ and $h$ are metrics on $M$ satisfying (1.1). We call such metrics *twin metrics* or *dual metrics*.

In this paper, starting from a manifold $M$ endowed with a metric $h = (h_{\mu\nu})$ and a symmetric linear connection $\Gamma = (\Gamma^{\alpha}_{\mu\nu})$, we obtain a $K$-structure as

$$K^{\alpha}_{\mu} = h^{\alpha\nu}S_{\mu\nu}$$

where $S_{\mu\nu} \equiv R_{(\mu\nu)}(\Gamma)$ is the symmetric part of the Ricci tensor of the given connection $\Gamma$. The general idea is the following. Let us first set $g_{\mu\nu} = R_{(\mu\nu)}(\Gamma)$; according to the results of our earlier paper [6] one can show that $g$ is in fact a new metric and that $h$ and $g$ are "twin metrics" if one assumes a suitable variational principle based on the action

$$A(\Gamma, h) = \int_M f(S)\sqrt{h}\, dx \tag{1.2}$$

and imposes independent variations over the metric and the connection. Here $f(S)$ is a given function of one real variable, which we assume to be analytic, while the scalar $S = S(\Gamma, h)$ is the Ricci square invariant

$$S = h^{\mu\alpha}h^{\nu\beta}S_{\alpha\beta}S_{\mu\nu}$$

If $f$ is a generic analytic function and $n > 2$ one gets either a (pseudo-) Riemannian almost-product structure or an almost-complex structure. In fact, as we shall see below (Theorem 2.4), the Euler-Lagrange equations for (1.2) are generically equivalent to the following system of equations for two metrics $h_{\mu\nu}$ and $g_{\mu\nu}$:

$$(h^{-1}g)^2 = \frac{c}{n}I \tag{1.3}$$

$$Ric(g) = g \tag{1.4}$$

where the real number $c$ is a root of the equation

$$f'(S)S - \frac{n}{4}f(S) = 0 \tag{1.5}$$

As an example, if one takes $f(S) = (nS + c(8-n))^2$, then $S = c$ is a solution of (1.5) if $n \neq 8$ and another (degenerate) solution is $S = c(n-8)/n$. Turning to the general discussion, if $c > 0$ then, as it was explained above, solutions of (1.3) - (1.4) are in one-to-one correspondence with almost-product Einstein manifolds $(M, g, P)$; while for $c < 0$ one gets anti-Hermitian Einstein manifolds $(M, g, J)$.

Before proceeding further let us explain why the action (1.2) is interesting and important in Mathematical Physics and especially in the theory of gravity.

As is well known gravitational Lagrangians which are nonlinear in the scalar curvature of a metric give rise to equations with higher (more than second) derivatives or to the appearance of additional matter fields [52, 40]. This strongly depends on having taken a metric as the only basic variable and the equations ensuing from such Lagrangians show an explicit dependence on the Lagrangian itself. An important example of a non-linear Lagrangian leading to equations with higher derivatives is given by Calabi's variational principle [11], which shall be discussed in a forthcoming paper.

It was shown in [22] that, in contrast, working in the first order (Palatini) formalism, i.e. assuming independent variations with respect to a metric and a symmetric connection, then, for a large class of Lagrangians of the form $f(R)$, where $R$ is the scalar curvature, the equations obtained are almost independent on the Lagrangian, the only such dependence being in fact encoded into constants (cosmological and Newton's ones). In this sense the equations obtained are "universal" and turn out to be Einstein equations in generic cases. Considering nonlinear gravitational Lagrangians which still generate Einstein equations is particularly important since they provide a simple but general approach to governing topology in dimension two [23] and in view of applications to string theory [57].

In a previous paper of ours [7] this discussion was extended to the case of Lagrangians with an arbitrary dependence on the square of the symmetrized Ricci tensor of a metric and a (torsionless) connection, finding roughly that the universality of Einstein equations also extends to this class of spacetimes. In this case, however, new important properties appear: as we have already mentioned above, depending in fact on the form of the Lagrangian and on the signature of the metric, one gets an almost-product Einsteinian structure or an almost-complex Einsteinian structure. Topological and geometrical obstructions for the global existence of a solution of the variational problem for this class of Lagrangians will be here considered in sections 4 and 5.

Recently there has been some interest on the problem of signature change

in General Relativity [20, 33, 17]; the non-standard signature $(10+2)$ has been considered also in superstring theory ($F$-theory [56]) and extra time-like dimensions in Kaluza-Klein theory are considered in [50, 2, 1]. Our results seem therefore to show new aspects of this problem which can be relevant also for quantum gravity. For a mathematical consideration of metrics with arbitrary signature which can be relevant to mathematical physics see for example [16, 32, 38, 4]

This paper is organized as follows. In the next section it will be shortly recalled how to get eqs. (1.3) and (1.4) from the action (1.2). In section 3 it is shown that eq. (1.3) can be always solved locally for any given metric $g$, in particular satisfying (1.4). In section 4 we discuss the $K$-structures. In section 5 we discuss the problems of the global existence as well as the classification of almost-product Einstein manifolds. In section 6 we prove that a real anti-Hermitian metric on a complex manifold satisfies the Kähler condition on the same manifold treated as a real manifold if and only if the metric is the real part of a holomorphic metric on this manifold. Finally, we consider also examples of almost-product Einstein manifolds and anti-Kählerian Einstein manifolds. Theorems of section 3 are proved in the Appendix.

## 2 Field Equations

In this section we shall present in a more geometrical form the results of our earlier paper [7], which form the basis of the further results presented hereafter. Consider, in the first order (Palatini) formalism, the family of actions

$$A(\Gamma, h) = \int_M f(S)\sqrt{h}\; dx \tag{2.1}$$

where: $M$ is a n-dimensional manifold ($n > 2$) endowed with a metric $h_{\mu\nu}$ and a torsionless (i.e., symmetric) connection $\Gamma^\sigma_{\mu\nu}$; the Lagrangian density is $L = f(S)\sqrt{h}$, where $f(S)$ is a given function of one real variable, which we assume to be analytic and $\sqrt{h}$ is a shorthand for $|\det(h_{\mu\nu})|^{\frac{1}{2}}$; the scalar $S$ is the symmetric part of Ricci square-invariant, considered as a first order scalar concomitant of a metric and (torsionless) connection, *i.e.*

$$S = S(h, \Gamma) = h^{\mu\alpha} h^{\nu\beta} S_{\alpha\beta} S_{\mu\nu} \tag{2.2}$$

being $S_{\mu\nu} = R_{(\mu\nu)}(\Gamma)$ the symmetric part of Ricci tensor, defined according to

$$R^\lambda_{\mu\nu\sigma}(\Gamma) = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} - \Gamma^\lambda_{\alpha\sigma}\Gamma^\alpha_{\mu\nu}$$

$$R_{\mu\sigma}(\Gamma) = R^{\nu}_{\mu\nu\sigma}(\Gamma) \quad (\alpha, \mu, \nu, ... = 1, ..., n)$$

Following [7] the Euler–Lagrange equations of the action (2.1) with respect to independent variations of $h$ and $\Gamma$ are

$$f'(S)h^{\alpha\beta}S_{\mu\alpha}S_{\nu\beta} - \frac{1}{4}f(S)h_{\mu\nu} = 0 \tag{2.3}$$

$$\nabla_\lambda(f'(S)\sqrt{h}\, h^{\mu\alpha}h^{\nu\beta}S_{\alpha\beta}) = 0 \tag{2.4}$$

where $\nabla_\lambda$ is the covariant derivative with respect to $\Gamma$. Transvecting (2.4) with $h^{\mu\nu}$ tells us that the scalar $S$ has to obey the following real analytic equation

$$f'(S)S - \frac{n}{4}f(S) = 0 \tag{2.5}$$

which allows to describe the general features of the non-linear system (2.3)-(2.4) and tells us, in turn, that $S$ is generically forced to be a constant. More precisely, it was shown in [7] that whenever (2.5) admits an (isolated) simple root $S = c$ then the system is "essentially equivalent" to Einstein equations for a new metric $g$ with a cosmological constant, in the precise sense which is hereafter described in greater detail. Consider in fact any solution

$$S = c \tag{2.6}$$

of eq. (2.5) and assume that $f'(c) \neq 0$. Then eq. (2.4) reduces to

$$\nabla_\lambda(\sqrt{h}h^{\mu\alpha}h^{\nu\beta}S_{\alpha\beta}) = 0 \tag{2.7}$$

while equation (2.3) reduces to

$$h^{\alpha\beta}S_{\mu\alpha}S_{\nu\beta} = \epsilon h_{\mu\nu} \tag{2.8}$$

where a new constant $\epsilon$ depending on $c$ arises according to the rule

$$\epsilon = f(c)/4f'(c) = c/n \tag{2.9}$$

From (2.8) the regularity condition

$$[\det(S_{\mu\nu})]^2 = \epsilon^n[\det(h_{\mu\nu})]^2 \tag{2.10}$$

follows, which entails in particular that $det(S_{\mu\nu}) \neq 0$ provided $\epsilon \neq 0$. Under this last hypothesis let $S^{\mu\nu}$ be the inverse matrix of $S_{\mu\nu}$ , so that from (2.8) we have:

$$h^{\mu\alpha}h^{\nu\beta}S_{\alpha\beta} = \epsilon S^{\mu\nu} \tag{2.11}$$

By using (2.10) and (2.11) we finally rewrite (2.7) as follows

$$\nabla_\lambda[\sqrt{|det(S_{\alpha\beta}(\Gamma))|}S^{\mu\nu}(\Gamma)] = 0 \tag{2.12}$$

which will be considered as a new equation in $\Gamma$.

Let us recall now the following well-known result, essentially due to Levi-Civita: for $n > 2$, any metric $g$ and any symmetric connection $\Gamma$ the general solution of the equation

$$\nabla_\alpha(\sqrt{g}\, g^{\mu\nu}) = 0 \tag{2.13}$$

considered as an equation for $\Gamma$ is the Levi–Civita connection $\Gamma = \Gamma_{LC}(g)$ *i.e.* locally

$$\Gamma^\sigma_{\mu\nu}(g) = \frac{1}{2}g^{\sigma\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \tag{2.14}$$

Therefore, the Ricci tensor $R_{\mu\nu}(\Gamma)$ of $\Gamma$ is automatically symmetric and in fact identical to the Ricci tensor $R_{\mu\nu}(g)$ of the metric $g$ itself.

We can then prove the following:

**Proposition 2.1.** *Let us assume that* $det(S_{\alpha\beta}) \neq 0$. *Then a connection* $\Gamma$ *satisfies eq. (2.12) if and only if there exists a metric* $g_{\mu\nu}$ *such that*

$$R_{\mu\nu}(g) = g_{\mu\nu} \tag{2.15}$$

*and* $\Gamma = \Gamma_{LC}(g)$ *is the Levi–Civita connection of* $g$.

*Proof.* Let $\Gamma$ be a connection satisfying eq. (2.12) and let us set

$$g_{\mu\nu} = S_{\mu\nu}(\Gamma) \tag{2.16}$$

The tensorfield $g$ is a metric due to the condition $det(S_{\alpha\beta}) \neq 0$. Then it follows that $\Gamma$ has to be the Levi–Civita connection of the metric $g$; moreover one has

$$S_{\mu\nu}(\Gamma) = R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(g) \tag{2.17}$$

so that (2.15) follows from (2.16) and (2.17).

Conversely, let us give a metric $g_{\mu\nu}$ satisfying (2.15) and let us take $\Gamma = \Gamma_{LC}(g)$. One has again relations (2.17). From (2.17) and (2.15) it follows then that (2.16) holds. Therefore $g^{\mu\nu} = S^{\mu\nu}(\Gamma)$ and $\det(S_{\mu\nu}(\Gamma)) = \det(g_{\mu\nu}) \neq 0$. Hence we see that eq. (2.14) reduces to (2.15), which is satisfied since $\Gamma = \Gamma_{LC}(g)$. Our claim is then proved. (Q.E.D.)

According to the previous discussion, we see that the Euler–Lagrange equations (2.3) and (2.4) are thence equivalent to the following equations for two metrics $h$ and $g$:

$$h^{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} = \epsilon h_{\mu\nu} \tag{2.18}$$

$$R_{\mu\nu}(g) = g_{\mu\nu} \tag{2.19}$$

which are in fact nothing but eqs. (1.3) and (1.4) of the Introduction. The relation between the system (2.18)-(2.19) and the Euler-Lagrange equations in the form (2.3)-(2.4) or (2.11)-(2.12) is given by setting $g_{\mu\nu} = R_{(\mu\nu)}(\Gamma)$.

We will now use the description of (pseudo-) Riemannian almost-product structures and almost-complex structures with a Norden metric in terms of a pair of metrics (twin or dual metrics). Let us consider a triple $(M, h, K)$ where $M$ is a differentiable manifold, $h$ is a metric on $M$ and $K$ is a (1,1) tensorfield on $M$ such that the following holds

$$K^2 = \epsilon I, \quad K^t h K = \epsilon h$$

As we said in the Introduction such a triple defines a (pseudo-)Riemannian almost-product structure if $\epsilon = 1$, while it defines an almost-complex structure with a Norden metric $h$ if $\epsilon = -1$. The triple $(M, h, K)$ admits an equivalent description as another triple $(M, h, g)$, where $g$ is a metric on $M$ satisfying the relation $(h^{-1}g)^2 = \epsilon I$ or the equivalent relation $(g^{-1}h)^2 = \epsilon I$, because of the following elementary proposition.

**Proposition 2.2.** *Let $K$ and $h$ be real $n \times n$ matrices and $\epsilon = +1$ or $-1$. Then the matrices $K$ and $h$ satisfy the relations*

$$h^t = h, \quad det\ h \neq 0, \quad K^2 = \epsilon I, \quad K^t h K = \epsilon h \tag{2.20}$$

*if and only if there exists a real matrix $g$ such that $g$ and $h$ satisfy the relations*

$$h^t = h, \quad det\ h \neq 0, \quad g^t = g, \quad (h^{-1}g)^2 = \epsilon I \tag{2.21}$$

*Moreover one has*

$$g = hK \tag{2.22}$$

*and*

$$(K^{-1})^2 = \epsilon I, \quad K^{-1t} g K^{-1} = \epsilon g \tag{2.23}$$

*Proof.* If one has (2.20) then define $g$ by (2.22) and check (2.21) and (2.23). Conversely, if one has (2.21) then define $K = h^{-1}g$ and check (2.20).

The claim is proved. (Q.E.D.)

We can therefore state the following theorem.

**Theorem 2.3.** *Let $M$ be a $n$-dimensional manifold, $n > 2$, with a metric $h$ and a symmetric connection $\Gamma$ and let us consider the Euler-Lagrange equations (2.3)-(2.4) for the action (2.1). Let us assume that the analytic function $f(S)$ is such that eq. (2.5) has an isolated root $S = c$ with $f'(c) \neq 0$; setting then $g_{\mu\nu} = R_{(\mu\nu)}(\Gamma)$, the Euler-Lagrange equations imply the relations $(h^{-1}g)^2 = \epsilon I$, $Ric(g) = g$. Therefore:*

*(i) if $\epsilon > 0$ after rescaling and denoting $P = g^{-1}h$ one gets an almost-product Einstein manifold $(M, g, P)$, i.e.*

$$Ric(g) = \gamma g$$

$$P^2 = I, \quad g(PX, PY) = g(X, Y), \quad X, Y \in \chi(M) \quad ;$$

*(ii) if $\epsilon < 0$ after rescaling and denoting $J = g^{-1}h$ one gets instead an anti-Hermitian Einstein manifold $(M, g, J)$, i.e.*

$$Ric(g) = \gamma g$$

$$J^2 = -I, \quad g(JX, JY) = -g(X, Y), \quad X, Y \in \chi(M)$$

*Here $\chi(M)$ is the Lie algebra of vector fields on $M$.*

Notice that the signatures of the metrics $h$ and $g$ in the almost-product case can be in principle arbitrary, while in the almost-complex case the signature is $(m, m)$. In any case they will be lower–semicontinuous functions, so that without any restriction we can assume that they remain constant in the neighborhood of a generic point; in particular they will not change in connected components of the manifold. In the next section we will therefore study the equation $(h^{-1}g)^2 = \epsilon I$ for a generic point on the manifold, i.e. study it algebraically as a matrix equation.

*Remark.* Notice that $\epsilon = 0$ corresponds to the case $S = 0$, which holds iff $f(0) = 0$. Then we have to distinguish two subcases $f'(0) = 0$ and $f'(0) \neq 0$. In the first subcase both equations (2.3) and (2.4) are automatically satisfied and no condition for $g$ and $\Gamma$ arise. Therefore any pair $(g, \Gamma)$ solves this subcase. When $f'(0) \neq 0$ then (2.3) is instead equivalent to the algebraic

equation $[h^{-1}S(\Gamma)]^2 = 0$, which leads to an *almost-tangent* structure [19], while (2.4) remains unchanged. We shall not discuss this case in the present paper.

# 3 Solutions of the Matrix Equations

Let us then consider the matrix equation

$$(h^{-1}g)^2 = \epsilon \mathbf{I}_n \tag{3.1}$$

where $h$ and $g$ are symmetric non-degenerate real $n \times n$ matrices, $\mathbf{I}_n$ is the identity matrix in $n$ dimensions and $\epsilon$ is a non-vanishing real number.

In order to solve equation (3.1) we first notice that it is manifestly $\mathrm{Gl}(n, I\!\!R)$–invariant under the canonical right-action $(h, g) \mapsto (A^t hA, A^t gA)$ where $A^t$ denotes the matrix transpose to $A$. More exactly, transforming $h$ and $g$ as metrics one can observe that $P = h^{-1}g$ transforms by a similarity transformation $P \to A^{-1}PA$ (i.e. as a (1,1) tensor). Equation (3.1) is also invariant under the transformation $(h, g, \epsilon) \mapsto (g, h, \epsilon^{-1})$. Moreover, we have $(\det(g)/\det(h))^2 = \epsilon^n$, so that when $n$ is even there are no restrictions on the sign of $\epsilon$, while $\epsilon$ has to be positive when $n$ is odd.

It is always possible to rescale $g$ by $\sqrt{\mid \epsilon \mid}$ and reduce (3.1) to the canonical form $(h^{-1}g)^2 = \pm \mathbf{I}_n$.

When $\epsilon$ is positive equation (3.1) admits always the trivial solution $g = \sqrt{\epsilon}\, h$; however, this does not exhaust all the possible solutions. Let us first observe in fact, by standard minimal-polynomial arguments (see e.g. [35]), that the matrix equation

$$P^2 = \mathbf{I}_n \tag{3.2}$$

admits only solutions of the form $P = M^{-1}D_k M$ for some non-singular matrix $M$, where the matrices $D_k$ (Jordan forms) are diagonal

$$D_k = \begin{pmatrix} -\mathbf{I}_k & 0 \\ 0 & \mathbf{I}_{n-k} \end{pmatrix} \tag{3.3}$$

and $k = 0, \ldots, n$. This result can be restated as follows: *if an automorphism $P$ of the vector space $I\!\!R^n$ satisfies (3.2) then there exists a basis in which $P$ is represented by one of the matrices $D_k$.* The non-negative integer $k$ is an invariant of this authomorphism. In fact such an automorphism $P$ ($P^2 = id$) represents an almost–product structure on $I\!\!R^n$. The set of all solutions of the equation $(h^{-1}g)^2 = \mathbf{I}_n$ is then described by the following

theorem (compare [35] Th. 4.5.15 case II b):

**Theorem 3.1.** *Let $g = g^t$ and $h = h^t$ be two real (symmetric) non-degenerate matrices (metrics). Then the following are equivalent:*

**a)** $(h^{-1}g)^2 = \mathbf{I}_n$

**b)** *the two metrics $h$ and $g$ are simultaneously diagonalizable with $\pm 1$ on the diagonal, i.e. there exists a real non-degenerate matrix $R$ such that*

$$h = R^t D_h R, \qquad g = R^t D_g R$$

*and $D_h$ and $D_g$ are diagonal matrices with $+1$ or $-1$ on the diagonal.*

The proof of this theorem is given in the Appendix.

Let us proceed to discuss the case $(h^{-1}g)^2 = -\mathbf{I}_n$. It is known that if $J$ is any $n \times n$ real matrix satisfying the relation

$$J^2 = -\mathbf{I}_n \tag{3.4}$$

then $n$ must be even, $n = 2m$, $J$ can be represented as

$$J = M J_o M^{-1}$$

where $J_o$ is the canonical form

$$J_o = \begin{pmatrix} 0 & \mathbf{I}_m \\ -\mathbf{I}_m & 0 \end{pmatrix} \tag{3.5}$$

and $M$ is a nondegenerate real matrix. In fact one deals with a complex structure and the matrix $J_o$ gives the canonical complex structure on $\mathbb{R}^{2m}$. The following holds true:

**Theorem 3.2.** *Let $h = h^t$ and $g = g^t$ be two $2m \times 2m$ real (symmetric) non-degenerate matrices (metrics). Then the following are equivalent:*

**a)** $(h^{-1}g)^2 = -\mathbf{I}_{2m}$

**b)** *there exists a real non-degenerate matrix $R$ such that*

$$h = R^t \begin{pmatrix} \mathbf{I}_m & 0 \\ 0 & -\mathbf{I}_m \end{pmatrix} R, \qquad g = R^t \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_m & 0 \end{pmatrix} R$$

*i.e. in the appropriate coordinate system the two metrics $g$ and $h$ take the following canonical forms*

$$K_h = \begin{pmatrix} \mathbf{I}_m & 0 \\ 0 & -\mathbf{I}_m \end{pmatrix}, \qquad K_g = \begin{pmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_m & 0 \end{pmatrix}.$$

Also this proof is given in the Appendix.

From theorems 3.1 and 3.2 it follows that locally for any given metric $g$ one can construct a twin metric $h$. If $g$ satisfies $Ric(g) = g$ this means that locally one produces an almost-product or an almost-complex Einsteinian structure.

# 4 $K$-Structures and Kähler-like Manifolds

We first present here a formalism which at once describes properties of various structures important in differential geometry, such as almost-complex and almost-product structures, Hermitian and anti-Hermitian metrics, Kähler manifolds and locally decomposable manifolds, *etc...*, and then, in the next Sections, consider in more detail the pseudo-Kählerian and anti-Käherian metrics.

Let $M$ be a smooth manifold, $TM$ its tangent bundle and $\chi(M)$ the algebra of vectorfields on $M$. A $(K, \epsilon)$-*structure* ($K$-*structure* in short) on $M$ is a field of endomorphisms $K$ on $TM$ such that $K^2 = \epsilon I$, where $\epsilon = \pm 1$. Thus $\epsilon = 1$ corresponds to an almost-product structure, while $\epsilon = -1$ provides an almost-complex structure.

Let $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ be a connection, denoted by $(X, Y) \to \nabla_X Y$. A $K$-structure is *integrable* iff there exists an a linear torsionless connection on $M$ such that $\nabla K = 0$; or equivalently the Nijenhuis tensor $N$

$$N(X, Y) = [KX, KY] - K[KX, Y] - K[X, KY] + \epsilon[X, Y]$$

vanishes. If $K$ is integrable then there exists an atlas of adapted coordinate charts on $M$ in which $K$ takes a canonical form (see *e.g.* [36]).

**Definition 1.** A 5-tuple $(M, K, g, \epsilon, \sigma)$ is called a $(K, g)$-*manifold* if $g$ is a metric on $M$ and $K$ is a $K$-structure, $K^2 = \epsilon I$, such that

$$g(KX, KY) = \sigma g(X, Y) \tag{4.1}$$

for all vectorfields $X$ and $Y$ on $M$. Here $\sigma = \pm 1$. In this case we shall say that the metric $g$ is *$K$-compatible* (or a *$K$-metric* in short).

The definition above unifies the following four cases: the case $\epsilon = 1, \sigma = 1$ corresponds to a (pseudo-) Riemannian almost-product structure; the case $\epsilon = 1, \sigma = -1$ provides an almost para-Hermitian structure; the case $\epsilon = -1, \sigma = 1$ is known as an almost-(pseudo)-Hermitian structure; and finally the case $\epsilon = -1, \sigma = -1$ corresponds to an almost-complex structure with a Norden metric.

Introduce a (0,2) tensorfield $h$, the *twin* of $g$, by

$$h(X, Y) = g(KX, Y) \tag{4.2}$$

Then

$$h(X, Y) = \epsilon\sigma h(Y, X), \qquad h(KX, KY) = \sigma h(X, Y) \tag{4.3}$$

Notice that for $\epsilon\sigma = 1$ the twin tensor is a metric (and this is, in fact, the case we have obtained from our variational principle), while for $\epsilon\sigma = -1$ the twin tensor is a two-form (and one deals with an almost-Hermitian or almost para-Hermitian structure).

Let $\psi$ be a $(0, 3)$ tensorfield defined by the formulae

$$\psi(X, Y, Z) = g((\nabla_X K)Y, Z) \equiv (\nabla_X h)(Y, Z) \tag{4.4}$$

In a coordinate language $\psi_{\alpha\mu\nu}$ is nothing but $\nabla_\alpha h_{\mu\nu}$. It possesses the following properties

$$\psi(X, Y, Z) = -\sigma\psi(X, KY, KZ) = \sigma\epsilon\psi(X, Z, Y) \tag{4.5}$$

Notice that the classification of almost-Hermitian structures [31], Riemannian almost-product structures [42] as well as almost-complex structures with a Norden metric [25] is based on algebraic properties of $\psi$: namely, one decomposes $\psi$ into irreducible components under the action of the appropriate group. The most restrictive class is Kähler-like, when simply $\psi = 0$. If the tensorfield $\psi$ vanishes then automatically $\nabla K = 0$ for a torsionless (Levi-Civita) connection and the Nijenhuis tensor $N$ is forced to vanish, too ([58, 59] c.f. also formulae (6.7)). Therefore the corresponding $K$-structure is integrable. It leads to the following definition:

**Definition 2.** A metric $g$ on a $(K, g)$-manifold is called a *Kähler-like metric* if $\nabla K = 0$, i.e.

$$\nabla_X(KY) = K\nabla_X Y, \quad X, Y \in \chi(M) \tag{4.6}$$

where $\nabla$ is the Levi-Civita connection of $g$ itself.

If $\epsilon = -1, \sigma = 1$ then a $K$-metric is called a Kähler metric. If $\epsilon = 1, \sigma = 1$ then a $K$-metric shall be called a *pseudo-Kählerian metric* (it is also called a (pseudo-) Riemannian locally decomposable metric). The case $\epsilon = 1, \sigma = -1$ shall be considered in a forthcoming publication [8] (see also [48, 4]). Our results on anti-Kählerian manifolds ($\epsilon = -1, \sigma = -1$) will be presented in sections 6 and 7. The following Proposition extends the Proposition 3.6 in [37] to an arbitrary $(K, g, \epsilon, \sigma)$-structure

**Proposition 4.1.** *The Riemann curvature $R(X,Y)Z$ and the Ricci tensor $S(X,Y)$ of a Kähler-like manifold (M,K,g) satisfy the following properties:*

$$R(X,Y)\circ K = K\circ R(X,Y), \qquad R(KX,KY) = \sigma R(X,Y) \tag{4.7}$$

$$S(KX,KY) = \sigma S(X,Y), \quad (\sigma - \epsilon)S(X,Y) = tr[V \mapsto K(R(X,KY)V)] \tag{4.8}$$

*Proof:* The proof is a simple repetition of the proof of Proposition 3.6 in [37], provided one suitably takes into account formulae (4.3). (Q.E.D.)

Consider now the twin $F$ of the Ricci tensor $S$

$$F(X,Y) = S(KX,Y) \tag{4.9}$$

Then

$$F(X,Y) = \epsilon\sigma F(Y,X), \quad F(KX,KY) = \sigma F(X,Y) \tag{4.10}$$

Notice that the symmetry property of $F$ is exactly the same as for $h$. Therefore, we conclude the following:

**Lemma 4.2.** *A Kähler-like manifold is Einstein iff the dual of $S$ is proportional to the dual of $g$, i.e. $F(X,Y) \sim h(X,Y)$.*

Notice that for $\epsilon\sigma = -1$ both twins $F$ and $h$ are two-forms. This Lemma

in the Kählerian case leads to a necessary condition on the first Chern class for a manifold to have an Einstein-Kähler metric [11, 60]. Recall the Goldberg conjecture [29] (see also [44, 51]) saying that almost-Kähler Einsteinian manifold is a complex one. It means that an Einstein almost-Hermitian manifold with a closed Kähler form is automatically Hermitian, *i.e.* its almost-complex structure is integrable. An extension of the Goldberg conjecture to the other Kähler-like manifolds will be disscuss elsewhere [8].

# 5 Almost Product Einstein Manifolds

In this section we consider the problems of the global existence and classification of almost–product Einstein manifolds. At the begining we shall recall some basic facts about almost-product and (pseudo-) Riemannian almost-product structures.

The simplest examples of almost-product structures are product manifolds, *i.e.* manifolds which are the Cartesian product of two manifolds

$$M = M_1 \times M_2 \tag{5.1}$$

In this case the tangent bundle splits as $TM = TM_1 \oplus TM_2$ and $P = P_2 - P_1$, where $P_i$ are the corresponding projections on $TM_i$, $i = 1, 2$. More generally, giving an almost-product structure is equivalent to splitting the tangent bundle into two complementary subbundles (*distributions* or *almost-foliations*): $TM = V \oplus H$; in this case $P = P_V - P_H$. $P$ is integrable iff both the distributions are integrable. (*i.e.* they are *foliations*). Integrable almost-product structures are also called *locally product manifolds* [58] since locally they have the form (5.1). It means that locally (around any point), there exists an adapted coordinate system $(x^a, y^\alpha)$, $a = 1, ..., k$ and $\alpha = 1, ..., n-k$, such that the tensorfield $P$ takes the canonical form (3.3), *i.e.* $P\partial_a = -\partial_a$ and $P\partial_\alpha = \partial_\alpha$.

Similarly, one can consider other structures: for example, a (pseudo-) Riemannian product manifold as a product of two (pseudo-) Riemannian manifolds

$$(M, g) = (M_1, g_1) \times (M_2, g_2) \tag{5.2}$$

where $g = g_1 \oplus g_2$. More generally, an almost-product (pseudo-) Riemannian structure $(M, P, g)$ is integrable iff $\nabla^g P = 0$ for the Levi–Civita connection $\nabla^g$ of $g$. In this case we speak of a *locally decomposable* (pseudo-)Riemannian manifold; in the present note we shall however propose to call it a *pseudo-Kähler manifold.* For locally decomposable (pseudo-) Riemannian structures

both foliations are *totally geodesic* [58, 59]. In an adapted coordinate system the metric $g$ "separates the variables" (see [58, 59])

$$ds^2 = g_{ab}(x)dx^a dx^b + g_{\alpha\beta}(y)dy^\alpha dy^\beta \tag{5.3}$$

The twin metric has the form $h = h_1 \ominus h_2$, *i.e.*

$$ds_h^2 = g_{ab}(x)dx^a dx^b - g_{\alpha\beta}(y)dy^\alpha dy^\beta$$

Since $\nabla^g_\mu h_{\alpha\beta} \equiv \psi_{\mu\alpha\beta} = 0$ then we have in this case $\Gamma(g) = \Gamma(h)$. In fact, this property gives an equivalent definition of pseudo-Kähler manifolds, provided $g$ and $h$ are twin metrics.

Recall that an almost-product Einstein manifold is a triple $(M, g, P)$ where $g$ is a metric and $P$ is a $(1,1)$ tensorfield which satisfy

$$Ric(g) = \gamma g \tag{5.4}$$

$$P^2 = I, \quad g(PX, PY) = g(X, Y), \quad X, Y \in \chi(M) \tag{5.5}$$

Notice that $P = I$ and $P = -I$ give trivial examples of an almost-product structure. Non-trivial examples are given by the following:

**Proposition 5.1.** *Let $(M^n, g)$ be an Einstein manifold satisfying (5.4) with an indefinite metric $g$ of signature $q$, $1 \leq q < n$ . Then there exists on $M^n$ a non-trivial almost-product structure $P$ satisfying (5.5) and therefore one gets an almost-product Einstein manifold $(M, g, P)$.*

*Proof:* If $g$ is a pseudo-Riemannian metric on $M$ then it was proved in [13] that there exist a (strictly) Riemannian metric $h$ and an almost product structure $P$ on $M$ such that $g(X, Y) = h(PX, Y)$ and $h(PX, PY) = h(X, Y)$ for all vectorfields $X$ and $Y$ on $M$. Therefore one gets the relations (5.5). The almost-product structure $P$ is non-trivial since if $P = \pm I$ then $g = \pm h$; but in fact the metric $g$ is pseudo-Riemannian while $h$ is strictly Riemannian. (Q.E.D.)

It follows from the proposition above that any manifold with a strictly pseudo-Riemannian Einstein metric serves as an example of a pseudo-Riemannian almost-product Einsteinian manifold.

It should also be noted that construction of $P$ for a given pseudo-Riemannian metric $g$ is not a canonical one (and, in fact, depends on a choice of some "background" Riemannian metric on $M$). Therefore, a single

(possibly Einstein) pseudo-Riemannian metric leads, in principle, to several almost-product (Einsteinian) manifolds. It makes a striking difference between the solutions of our variational problem corresponding to the positive roots of the fundamental equation (2.5) and those corresponding to the negative roots. In the second case, as we shall see in the next Section, there are further topological obstructions for the existence of an almost-complex structure.

Let M be a pseudo-Kähler manifold, *i.e.* (locally) in adapted coordinate systems $(x^a, y^\alpha)$ the metric $g$ splits as $g = g_1 \oplus g_2$, where $g_{ab} = g_1(x)$ and $g_{\alpha\beta} = g_2(y)$. If both metrics in (5.3) are Einstein

$$R_{ab}(g_1) = \gamma g_{ab}, \quad R_{\alpha\beta}(g_2) = \gamma g_{\alpha\beta} \tag{5.6}$$

for the same constant $\gamma$, then it follows:

$$R_{AB}(g) = \gamma g_{AB} \tag{5.7}$$

Therefore, one has

**Proposition 5.2.** *A pseudo-Kählerian manifold is Einstein iff in any adapted coordinates $(x^a, y^\alpha)$ both metrics are Einstein for the same constant $\gamma$.*

*Proof:* See [58, 59]. (Q.E.D.)

Interesting examples of locally product (pseudo-) Riemannian manifolds which are not locally decomposable are given by *warped product* spacetimes [45, 18, 12]. Given two (pseudo-) Riemannian manifolds $(M_i, g_i)$, $i = 1, 2$, and a smooth function $\theta : M_1 \longrightarrow I\!R$, on the product manifold $M = M_1 \times M_2$ put the metric $g = g_1 \oplus e^{2\theta} g_2$. The resulting (pseudo-) Riemannian manifold $M = M_1 \times_\theta M_2$ is called a *warped product* manifold. It is of course an almost-product (pseudo-) Riemannian manifold and it is conformal to a locally decomposable one. It is an interesting and intriguing fact that many exact solutions of Einstein equations (including *e.g.* Schwarzschild, Robertson-Walker, Reissner-Nordström, de Sitter *etc...*) and also p-brane solutions ([3]) are, in fact, warped product spacetimes [12]. Therefore, these exact solutions provide beautiful examples of almost-product Einsteinian manifolds.

There are topological restrictions on a (paracompact) manifold $M$ for the existence of an almost-product structure of rank $k$, which are the same as for the existence of a (strictly) pseudo-Riemannian metric of signature

$(k, n-k)$, which are again the same as for the existence of a $k$-dimensional distribution. For example, for the existence of a metric with Lorentz signature on a compact manifold $M$ (*i.e.* for the existence of a nowhere vanishing vectorfield) the necessary and sufficient condition is that the Euler characteristic number vanishes.

# 6 Anti-Kählerian Manifolds

Now we consider in some detail the case of a $K$-metric with $\epsilon = -1, \sigma = -1$, which we call an anti-Kählerian metric.

**Definition 3.** A triple $(M, g, J)$, where $J$ is an almost-complex structure and the metric $g$ is anti-Hermitian: $g(JX, JY) = -g(X, Y), X, Y \in \chi(M)$ is called an *anti-Kählerian manifold* if $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection.

We will prove that a real anti-Hermitian metric on a complex manifold satisfies the Kähler condition $\nabla J = 0$ on the same manifold treated as a real manifold if and only if the metric is the real part of a holomorphic metric on this manifold.

Let $(M, J)$ be a $2m$-dimensional almost-complex real manifold and let $g$ be an anti-Hermitian metric on $M$. We extend $J$, $g$ and the Levi-Civita connection $\nabla$ by $\mathbb{C}$-linearity to the complexification of the tangent bundle $T_C M = T_M \otimes \mathbb{C}$. We use the same notation for the complex extended $g, J$ and $\nabla$. Then the Levi-Civita connection is the mapping $(X, Y) \to \nabla_X Y$ where $X$ and $Y$ are now complex vectorfields (*i.e.*, sections of $T_C M$). Then the (complex extended) torsion tensor $T$ vanishes

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and the ordinary formulae are valid for the connection:

$$\nabla_X g(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \tag{6.1}$$

and for the Riemann tensor

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{6.2}$$

where $X, Y, Z$ are complex vectorfields. For the sake of clarity, we stress that for the moment we are just complexifying the tangent bundle but we

do not assume the almost-complex structure $J$ is integrable. Let us now fix a (real) basis $\{X_1, ..., X_m, JX_1, ..., JX_m\}$ in each tangent space $T_xM$; then the set $\{Z_a, Z_{\bar{a}}\}$, where $Z_a = X_a - iJX_a$, $Z_{\bar{a}} = X_a + iJX_a$, forms a basis for each complexified tangent space $T_xM \otimes \mathbb{C}$. Unless otherwise stated, little Latin indices $a, b, c, ...$ run from 1 to $m$, while Latin capitals $A, B, C, ...$ run through $1, ..., m, \bar{1}, ..., \bar{m}$; for notational convenience we shall also bar capital indices and we shall assume $\bar{\bar{A}} = A$. One has $JZ_a = iZ_a$ and $JZ_{\bar{a}} = -iZ_{\bar{a}}$. We set $g_{AB} = g(Z_A, Z_B) = g_{BA}$. Then the following holds:

**Proposition 6.1.** *Let $(M, J)$ be an almost-complex manifold and $g$ be an anti-Hermitian metric on it. Then the complex extended metric $g$ (in the complex basis constructed above) satisfies the following conditions*

$$g_{a\bar{b}} = g_{\bar{b}a} = 0 \tag{6.3}$$

$$g_{\bar{A}\bar{B}} = \bar{g}_{AB} \tag{6.4}$$

*Proof.* Since the metric $g$ is anti-Hermitian, we have

$$g(Z_a, Z_{\bar{b}}) = -g(JZ_a, JZ_{\bar{b}}) = -g(iZ_a, -iZ_{\bar{b}}) = -g(Z_a, Z_{\bar{b}})$$

Therefore $g_{a\bar{b}} = 0$, which proves (6.3). The proof of (6.4) is well known. In fact we have

$$g_{\bar{a}\bar{b}} = g(Z_{\bar{a}}, Z_{\bar{b}}) = g(X_a + iJX_a, X_b + iJX_b)$$

$$= g(X_a, X_b) - g(JX_a, JX_b) + ig(JX_a, X_b) + ig(X_a, JX_b)$$

and

$$g_{ab} = g(X_a - iJX_a, X_b - iJX_b)$$

$$= g(X_a, X_b) - g(JX_a, JX_b) - ig(JX_a, X_b) - ig(X_a, JX_b)$$

Therefore we get $g_{\bar{a}\bar{b}} = \bar{g}_{ab}$. Similarly we consider the other components of $g_{AB}$ and hence we prove (6.4). (Q.E.D.)

It is customary to write a metric satisfying (6.3) and (6.4) as

$$ds^2 = g_{ab}dz^a dz^b + g_{\bar{a}\bar{b}}dz^{\bar{a}}dz^{\bar{b}} \tag{6.5}$$

We define now the complex Christoffel symbols $\Gamma^C_{AB}$ as

$$\nabla_{Z_A} Z_B = \Gamma^C_{AB} Z_C \tag{6.6}$$

It is known [58, 37] that if $\nabla J = 0$ then the torsion $T$ and the Nijenhuis tensor $N$ satisfy the identity

$$T(JX, JY) = \frac{1}{2} N(X, Y) \tag{6.7}$$

for any vectorfields $X$ and $Y$. Since the complex extended Levi-Civita connection $\nabla$ has no torsion, the complex Christoffel symbols are symmetric. In this case the complex structure $J$ is integrable so that the real manifold $M$ inherits the structure of a complex manifold. Let us now recall (see *e.g.* [37]) that there is a one-to-one correspondence between complex manifolds and real manifolds with an integrable complex structure. This means that there exist real, adapted (local) coordinates $(x^1, ..., x^m, y^1, ..., y^m)$ such that

$$J(\frac{\partial}{\partial x^a}) = \frac{\partial}{\partial y^a}, \quad J(\frac{\partial}{\partial y^a}) = -\frac{\partial}{\partial x^a}$$

Setting $z^a = x^a + iy^a$ and taking $X_a = \partial/\partial x^a$ one gets

$$Z_a = X_a - iJX_a = 2\partial/\partial z^a = 2\partial_a, \quad Z_{\bar{a}} = X_a + iJX_a = 2\partial/\partial \bar{z}^a = 2\partial_{\bar{a}}$$

where $\partial_A = \partial/\partial z^A$ and $z^{\bar{a}} = \bar{z}^a$. It appears that $z^a$'s form a complex (analytic) coordinate chart on $M$. Now from (6.1) one gets

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD}(Z_A g_{DB} + Z_B g_{DA} - Z_D g_{AB}) = g^{CD}(\partial_A g_{DB} + \partial_B g_{DA} - \partial_D g_{AB}) \tag{6.8}$$

Notice that the relation (6.1) is valid for the complex extended metric $g$ and complex vector fields $X, Y, Z$ if and only if it is valid for real vectorfields.

**Theorem 6.2.** *Let $M$ be a $m$-dimensional complex manifold, thought as a real $2m$-dimensional manifold with a complex structure $J$. Let us further assume that $M$ is provided with an anti-Hermitian metric $g$. We extend $J$, $g$ and the Levi-Civita connection $\nabla$ by $\mathbb{C}$-linearity to the complexified tangent bundle $T_C M$. Then the following conditions are equivalent:*

*(i)*

$$\nabla_X(JY) = J\nabla_X Y \tag{6.9}$$

*where $X$ and $Y$ are arbitrary real vectorfields;*

*(ii) the (complex) Christoffel symbols satisfy*

$$\Gamma^C_{AB} = 0 \quad \textit{except for} \;\; \Gamma^c_{ab} \;\; \textit{and} \;\; \Gamma^{\bar{c}}_{\bar{a}\bar{b}} = \bar{\Gamma}^c_{ab} \tag{6.10}$$

*(iii) there exists a local complex coordinate system $(z^1, ..., z^m)$ on $M$ such that the components of the complex extended metric $g_{ab}$ in the canonical form (6.5) are holomorphic functions*

$$\partial_{\bar{c}}\, g_{ab} = 0 \tag{6.11}$$

*Proof.* From (6.6) we have

$$\bar{\Gamma}^{C}_{AB} = \Gamma^{\bar{C}}_{\bar{A}\bar{B}}$$

The connection satisfies the conditions

$$\nabla_{Z_B}(JZ_c) = J\nabla_{Z_B}Z_c = i\nabla_{Z_B}(Z_c)$$
$$\nabla_{Z_B}(JZ_{\bar{c}}) = J\nabla_{Z_B}Z_{\bar{c}} = -i\nabla_{Z_B}(Z_{\bar{c}})$$

if and only if

$$\Gamma^{a}_{B\bar{c}} = \Gamma^{\bar{a}}_{Bc} = 0 \tag{6.12}$$

This proves the equivalence between $(i)$ and $(ii)$. Then for the Christoffel symbols (6.8) by taking (6.3) into account one gets

$$\Gamma^{a}_{b\bar{c}} = g^{aD}(\partial_b g_{D\bar{c}} + \partial_{\bar{c}} g_{Db} - \partial_D g_{D\bar{c}}) = g^{ad}\partial_{\bar{c}} g_{bd} \tag{6.13}$$

and from (6.12) it follows that

$$\partial_{\bar{c}} g_{bd} = 0 \tag{6.14}$$

The other relations (6.12) also are reduced to (6.14) or its complex conjugated. Therefore the relation (6.14) is equivalent to (6.11). This proves the equivalence between $(i)$ and $(iii)$. Our claim is thence proved. (Q.E.D.)

We have proved that a real anti-Hermitian metric on a complex manifold satisfies the Kähler condition $\nabla J = 0$ on the same manifold treated as a real manifold if and only if the metric is the real part of a holomorphic metric on this manifold. Therefore, there exists a one-to-one correspondence between anti-Kähler manifolds and complex Riemannian manifolds with a holomorphic metric as they were defined in [39] (see also [26]).

From an algebraic viewpoint, let us mention that we have been dealing with the following construction. Let $V$ be a real vector space with a complex structure $J$ and let $G$ be a complex-valued bilinear form on $V$. Let us set

$$F(X, Y) = G(X, Y) - G(JX, JY) - iG(X, JY) - iG(JX, Y)$$

Then we have

$$F(JX, JY) = -F(X, Y)$$

Now one takes the real (or imaginary) part of $F$ to get a real anti-Hermitian bilinear form on $V$.

## 7 Anti Kählerian Einstein manifolds

In this section we consider the problems of the global existence and classification of anti-Hermitian Einstein manifolds. Recall that an anti-Hermitian Einstein manifold is a triple $(M, g, J)$ where $g$ is a metric and $J$ is a $(1,1)$ tensorfield which satisfy

$$Ric(g) = \gamma g \tag{7.1}$$

$$J^2 = -I, \quad g(JX, JY) = -g(X, Y), \quad X, Y \in \chi(M) \tag{7.2}$$

Then the metric $g$ has necessarily the signature $(m, m)$ (see section 3), being $2m = \dim M$. Let us show that by taking the real part of a holomorphic Einstein metric on a complex manifold of complex dimension $m$ one can get a real Einstein manifold of real dimension $2m$.

From (4.8) we have for the Ricci tensor

$$Ric(g)(JX, JY) = -Ric(g)(X, Y) \quad X, Y \in \chi(M)$$

Therefore, analogously to (6.3), we have

$$R_{a\bar{b}} = 0 \tag{7.3}$$

We shall not attempt here to consider solutions of Einstein equations for a generic metric of the form (6.5) but consider only the case when $g_{ab}$ is a holomorphic function

$$\partial_{\bar{c}}\, g_{ab} = 0 \tag{7.4}$$

From (7.4) and (4.7) we get for the Riemann tensor:

$$R^{D}_{ABC} = 0 \quad \text{except for} \quad R^{d}_{abc} \quad \text{and} \quad R^{\bar{d}}_{\bar{a}\bar{b}\bar{c}} = \bar{R}^{d}_{abc} \tag{7.5}$$

The (complex) Einstein equations

$$R_{AB}(g) = \gamma g_{AB} \tag{7.6}$$

are thence equivalent to a pair of equations

$$R_{ab}(g_{cd}) = \gamma g_{ab} \tag{7.7a}$$

$$R_{\bar{a}\bar{b}}(g_{\bar{c}\bar{d}}) = \gamma g_{\bar{a}\bar{b}} \tag{7.7b}$$

To get a real solution of Einstein equations (7.1) from (7.7) one uses real coordinates $(x^\mu), \mu = 1, ..., 2m$ on $M$, i.e. $z^a = x^a + ix^{m+a}, a = 1, ..., m$ and writes the metric (6.5) as

$$ds^2 = g_{ab}dz^a dz^b + g_{\bar{a}\bar{b}}dz^{\bar{a}}dz^{\bar{b}} = g_{\mu\nu}dx^\mu dx^\nu \tag{7.8}$$

where $g_{\mu\nu}$ is a real metric. We have thence proved the following theorem:

**Theorem 7.1.** *If $(M, g, J)$ is an anti-Kählerian manifold, i.e. a complex manifold of complex dimension $m$ with a holomorphic metric $g_{ab}(z), a, b = 1, ..., m$ and a real metric $g_{\mu\nu}(x), \mu, \nu = 1, ..., 2m$ defined by (7.8), then the holomorphic metric $g_{ab}(z)$ satisfies (7.7a) if and only if the real metric $g_{\mu\nu}(x)$ is a solution of the Einstein equations (7.1)*

$$R_{\mu\nu}(g) = \gamma g_{\mu\nu} \tag{7.9}$$

As an example one can take a complex analytic continuation of any real analytic solution of Einstein equations. A simple example is

$$ds^2 = dz^a dz^a + \frac{(z^a dz^a)^2}{1 - z^a z^a} + \text{ complex conj.} = g_{\mu\nu}dx^\mu dx^\nu \tag{7.10}$$

This metric $g_{\mu\nu}$ on "the complex sphere": $w_1^2 + ... + w_{m+1}^2 = 1$ (which can be interpreted as a quadric $\zeta_1^2 + ... + \zeta_{m+1}^2 - \zeta_{m+2}^2 = 0$ in $\mathbb{C}P^{m+1}$ if one takes $w_i = \zeta_i/\zeta_{m+2,}$) gives a solution of the Einstein equations (7.8) and provides an example of an anti-Hermitian Einstein manifold $(M, g, J)$.

In particular for $m = 2$ we get a real solution of Einstein equations on the 4-dimensional real manifold $(w_1^2 + w_2^2 + w_3^2 = 1, w_i \in \mathbb{C})$ with a metric of signature $(+ + --)$.

Notice also that any Einstein metric on a compact Riemannian manifold $M^n$ leads to an anti-Kählerian Einstein metric on another real manifold $\mathcal{M}^{2n}$. It follows from the known fact [10] that any Einstein metric is analytic in a certain atlas on $M^n$. Therefore there exists a complex analytic continuation of the metric to a complex manifold of complex dimension $n$ which is a real anti-Kählerian manifold $\mathcal{M}^{2n}$.

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## Appendix

*Proof of Theorem 3.1*: The proof $b) \Rightarrow a)$ follows obviously from the fact that each two diagonal matrices with $\pm 1$ on their diagonals do satisfy our equation, which is invariant under the appropriate transformation.
The converse $a) \Rightarrow b)$ is less obvious. As we already know there exists a real non-degenerate matrix $M$ such

$$h^{-1}g = MD_kM^{-1} \tag{A.1}$$

From this one gets

$$\tilde{g} = \tilde{h}D_k$$

where $\tilde{g} = M^tgM$ and $\tilde{h} = M^thM$. Since $\tilde{g}$, $\tilde{h}$ and $D_k$ are symmetric matrices one has thence

$$\tilde{h}D_k = D_k\tilde{h} \tag{A.2}$$

Let us now represent $\tilde{h}$ in block–form:

$$\begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12}^t & \tilde{h}_{22} \end{pmatrix}$$

where $\tilde{h}_{11}^t = \tilde{h}_{11}$, $\tilde{h}_{22}^t = \tilde{h}_{22}$ and $\tilde{h}_{11}$ is a $k \times k$ matrix. Then from (A.2) we obtain

$$\begin{pmatrix} \tilde{h}_{11} & -\tilde{h}_{12} \\ \tilde{h}_{12}^t & -\tilde{h}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ -\tilde{h}_{12}^t & -\tilde{h}_{22} \end{pmatrix}$$

Therefore $\tilde{h}_{12} = 0$ and one gets

$$\tilde{h} = \begin{pmatrix} \tilde{h}_{11} & 0 \\ 0 & \tilde{h}_{22} \end{pmatrix}, \qquad \tilde{g} = D_k\tilde{h} = \begin{pmatrix} \tilde{h}_{11} & 0 \\ 0 & -\tilde{h}_{22} \end{pmatrix} \tag{A.3}$$

Now we make use of the fact that any real nondegenerate symmetric matrix is $^t$-congruent to a diagonal matrix whose diagonal elements are equal to $+1$ or $-1$, i.e.: $h_{11} = S_1^t D_{h_{11}} S_1$ (and analogously $h_{22} = S_2^t D_{h_{22}} S_2$), where $D_{h_{11}}$ (resp. $D_{h_{22}}$) is a diagonal matrix with $\pm 1$ along the diagonal. Therefore one has

$$\tilde{h} = S^t \begin{pmatrix} D_{h_{11}} & 0 \\ 0 & D_{h_{22}} \end{pmatrix} S, \qquad \tilde{g} = S^t \begin{pmatrix} D_{h_{11}} & 0 \\ 0 & -D_{h_{22}} \end{pmatrix} S$$

where $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$. Notice that $S$ commutes with $D_k$, i.e. $SD_k = D_k S$. Taking then $R = SM$ the theorem is proved. (Q.E.D.)

*Proof of Theorem 3.2*: To prove $b) \Rightarrow a)$ check that $K_h^{-1} K_g = J_o$ and then make use of the appropriate transformation properties.

In order to prove the converse $a) \Rightarrow b)$, in full analogy with the previous case one makes use of the fact that $h^{-1} g = M J_o M^{-1}$. Now this leads to the condition (with the same notation) :

$$\tilde{h} J_o = -J_o \tilde{h} \qquad (A.4)$$

Writing then $\tilde{h}$ in block–form:

$$\tilde{h} = \begin{pmatrix} a & b \\ b^t & d \end{pmatrix}$$

one has from (A.4):

$$\begin{pmatrix} -b & a \\ -d & b^t \end{pmatrix} = \begin{pmatrix} b^t & d \\ -a & -b \end{pmatrix}$$

Therefore $a = -d$, $b = b^t$ and we have

$$\tilde{h} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad (A.5a)$$

$$\tilde{g} = \tilde{h} J_o = \tilde{h} = \begin{pmatrix} -b & a \\ a & b \end{pmatrix} \qquad (A.5b)$$

Let us further show that there exists a real matrix $S$ such that

$$S J_o = J_o S \qquad (A.6)$$

$$\tilde{h} = S^t K_h S \tag{A.7a}$$

$$\tilde{g} = S^t K_g S \tag{A.7b}$$

To be sure that (A.6) holds take the matrix $S$ in the form

$$S = \begin{pmatrix} s & u \\ -u & s \end{pmatrix}$$

where $s$ and $u$ are real $m \times m$ matrices to be determined from the conditions (A.7a) and (A.7b). Equation (A.7a) reads as

$$\begin{pmatrix} s^t s - u^t u & s^t u + u^t s \\ u^t s + s^t u & u^t u - s^t s \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \tag{A.8a}$$

while the equation (A.7b) gives

$$\begin{pmatrix} -u^t s - s^t u & s^t s - u^t u \\ s^t s - u^t u & s^t u + u^t s \end{pmatrix} = \begin{pmatrix} -b & a \\ a & b \end{pmatrix} \tag{A.8b}$$

Notice that eq. (A.8a) is equivalent to (A.8b) so that we are left with the following conditions

$$s^t s - u^t u = a, \qquad s^t u + u^t s = b$$

This last equation can be rewritten in the complex form

$$(s + iu)^t (s + iu) = a + ib$$

Now it is known (see for example [35]) that any nondegenerate symmetric complex matrix $a + ib$ can be represented in the form

$$a + ib = N^t N$$

where $N$ is a (nondegenerate) complex matrix. Taking $s + iu = N$ and $R = SM^{-1}$ the theorem is proved. (Q.E.D.)

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